9. Determinants

- Determinant of a square matrix A is denoted by $|A|$ or det (A).
- Determinant of a matrix $A = [a]_{1\times 1}$ is $|A| = |a| = a$
- Determinant of a matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is given by, $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$
- Determinant of a matrix $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$ is given by (expanding along R_1):

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = (-1)^{1+1} a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + (-1)^{1+2} a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + (-1)^{1+3} a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Similarly, we can find the determinant of A by expanding along any other row or along any column.

- If A is a square matrix, then $A\,(adj\ A) = (adj\ A)\,A = |A|\,I$
- A square matrix A is said to be singular, if A=0
- A square matrix A is said to be non-singular, if $|A| \neq 0$
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- If A and B are square matrices of same order, then $|AB| = |A||B|$

Therefore, if A and B are non-singular matrices of same order, then AB and BA are also non-singular matrices of same order.

- If A is a non-singular matrix of order n, then $|adj\ A| = |A|^{n-1}$
- A square matrix A is invertible, if and only if A is non-singular and inverse of A is given by the formula:

$$A^{-1} = \frac{1}{|A|}(adj A)$$

- The various properties of determinants are as follows:
  - If the rows and the columns of a square matrix are interchanged, then the value of the determinant remains unchanged.

Example:

$$|A| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

This property is same as saying, if A is a square matrix, then $|A| = |A'|$

- If we interchange any two rows (or columns), then sign of determinant changes.

  Example:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix}, \text{ by applying } C_1 \leftrightarrow C_2$$

$$= \begin{vmatrix} b_3 & a_3 & c_3 \\ b_2 & a_2 & c_2 \\ b_1 & a_1 & c_1 \end{vmatrix}, \text{ by applying } R_1 \leftrightarrow C_3$$

- If any two rows or any two columns of a determinant are identical or proportional, then the value of the determinant is zero.

  Example:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ ka_1 & kb_1 & kc_1 \end{vmatrix} = 0, \text{ where } k \text{ is a constant}$$

- If each element of a row or a column of determinant is multiplied by a constant a, then its determinant value gets multiplied by a.

  Example:

• For any four numbers, a, b, c and d, the value $ad - bc$ can be represented as $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$. This type of representation of numbers or variables is called **determinant**. It is a determinant of order two.

• **Cramer's rule:**

For two simultaneous equations, $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$ where a_1, a_2, b_1, b_2, c_1 and c_2 are real numbers such that $a_1b_2 - a_2b_1 \neq 0$ and x and y are variables then:

$$x = \frac{D_x}{D} \text{ and } y = \frac{D_y}{D}$$

Here, $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, $D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$ and $D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$

Consistency of Three Equations in Two Variables

A system of equations is consistent if it has at least one solution, that is, a unique solution or an infinite solution. On the other hand, a system of equations is inconsistent if it has no solution.

A system of equations is consistent if the values of x and y obtained from any two equations satisfy the third equation. A system of three equations in two variables is consistent if they have the same solution.

Consider the following system of equations:

$a_1x + b_1y + c_1 = 0$
$a_2x + b_2y + c_2 = 0$
$a_3x + b_3y + c_3 = 0$
The necessary condition for the consistency of the given system of equations is
a1 b1 c1a2 b2 c2a3 b3 c3=0.

- Area of a triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) is given by,

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Since area is always positive, we take the absolute value of the above determinant.